Linking Makinson and Kraus-Lehmann-Magidor preferential entailments

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Abstract. Twelve years ago, various notions of preferential entailment have been introduced. The main reference is a paper by Kraus, Lehmann and Magidor (KLM), one of the main competitor being a more general version defined by Makinson (MAK). These two versions have already been compared, but it is time to revisit these comparisons. Here are our main results: (1) These two notions are equivalent, provided that we restrict our attention, as done in KLM, to the cases where the entailment respects logical equivalence (on the left and on the right). (2) A serious simplification of the description of the fundamental cases in which MAK is equivalent to KLM, including a natural passage in both ways. (3) The two previous results are given for preferential entailments more general than considered in some texts, but they apply also to the original definitions and, for this particular case also, the models can be simplified.

1 INTRODUCTION

Here is one possible presentation of preferential entailments: We are given some knowledge, represented as a set of logical formulas. This set can be associated with various kinds of objects, providing its "semantics": it can be associated with its set of models, or equivalently in the propositional case, with the set of the complete theories which entail the formulas. Or it can be associated with the set of theories (complete or not) which entail the formulas. Then we are given a binary relation among these objects, and we keep only the objects which are "preferred" (meaning minimal) for this relation. We get a stronger set of formulas, deduced "by default": we get also all the formulas associated with this reduced set of objects. This allows to reason in a non monotonic way, since augmenting the knowledge may invalidate previous conclusions. Indeed, some objects may become minimal in the smaller set associated with the new knowledge. We can allow more flexibility by considering copies of models, or copies of theories, defining the relation among these sets of copies. We get then four kinds of preferential entailments, called KLM below, which have been introduced by Kraus, Lehmann and Magidor (1990) in [7] ([7] requires some conditions on the relation, but adding these conditions is straightforward in our results).

Makinson (1994, first version in 1989) has defined a more general version [8], called MAK here. An unstructured "semantics" is defined simply by a satisfaction relation from some set of objects to the set of formulas, without any condition. It is then useless to consider sets of these objects instead of singletons: its suffices to define directly as our starting set, the set of sets that we would want to consider. Also, since nothing prevents two different objects from being associated with the same set of formulas, it is useless to consider copies of objects. A drawback of this simple definition is that the notion of classical deduction is lost. The "entailment" can be highly non standard, departing from classical logic and natural ways of reasoning. We can e.g. deduce $A \wedge B$ without deducing A. As a bonus, we can consider as our starting logic basically any non classical logic.

If we want to compare the two notions, we must determine which non standard behavior we admit in the "semantics" defining the preferential entailment. If we want to extend KLM entailment in order to deal with any situation accessible to MAK entailment, all we have to do is to admit any unconstrained satisfaction relation between the set of states and the set of formulas. We can even restrict our attention to the simplest of the four cases, defining the relation directly on the set of the objects describing the semantics. Thus, the interesting point is the other direction. Thanks to recent results on preferential entailments, we show that, provided classical equivalence is respected, MAK entailment is equivalent to KLM entailment. Since it is clear that KLM entailment respects logical equivalence, equivalence between the two formalisms holds in all the cases where it is possible. We describe a simple subclass of MAK models, which includes all the cases where MAK entailment respects classical equivalence, and for which it is easy to describe a passage from MAK formulation to KLM formulation and back. We improve previous results obtained by Dix and Makinson (1992) in [4] and by Voorbraak (1993) in [13]. The description of the subclass of MAK is much simpler than in [4].

In Section 2 we introduce the notations and the logical prerequisite necessary for this text. In Section 3 we remind the definitions and main properties of the kind of preferential entailments considered here, giving our results in Section 4.

2 NOTATIONS AND FRAMEWORK

• $\mathbf{L},\ \varphi,\ \mathcal{T}$: We work in a propositional language $\mathbf{L}$, and we use the same denotation $\mathbf{L}$ for its set of formulas. Letters φ, ψ denote *formulas* (identified with their equivalence class). Letters $\mathcal{T}$ or $\mathcal{C}$ denote sets of formulas.

• $V,\ \mathbf{M},\ \mu,\ \mathcal{P}(E),\ \mu \models \cdots$: $V(\mathbf{L})$ (vocabulary), denotes a set of propositional symbols and μ denotes an *interpretation for* $\mathbf{L}$, identified with the subset of $V(\mathbf{L})$ that it satisfies. The *satisfaction relation* is denoted by $\models$, $\mu \models \varphi$ and $\mu \models \mathcal{T}$ being defined classically. For any set E, $\mathcal{P}(E)$ denotes the set of its subsets. The set $\mathcal{P}(V(\mathbf{L}))$ of the interpretations for $\mathbf{L}$ is denoted by $\mathbf{M}$. A *model* of $\mathcal{T}$ is an interpretation μ such that $\mu \models \mathcal{T}$. The sets of the models of $\mathcal{T}$ and φ are denoted respectively by $\mathbf{M}(\mathcal{T})$ and $\mathbf{M}(\varphi)$.

• $\mathcal{T} \models \cdots, Th(\mathcal{T}), \mathbf{T}$: $\mathcal{T} \models \varphi$ and $\mathcal{T} \models \mathcal{T}_1$ are defined classically. A *theory* is a subset of $\mathbf{L}$ closed for deduction, and $\mathbf{T}$ denotes the set $\{\mathcal{T} \subseteq \mathbf{L} \ / \ \mathcal{T} = Th(\mathcal{T})\}$ of the theories of $\mathbf{L}$.

• $\mathbf{M}_1 \models \cdots, Th(\mu), Th(\mathbf{M}_1)$: A theory $\mathcal{C} \in \mathbf{T}$ is *complete* if

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$\forall \varphi \in \mathbf{L}$, $\varphi \in \mathcal{C}$ iff $\neg\varphi \notin \mathcal{C}$. The set $Th(\mu) = \{\varphi \in \mathbf{L} \ / \ \mu \models \varphi\}$ of the formulas satisfied by μ is the *theory of* μ. For any subset $\mathbf{M}_1$ of $\mathbf{M}$, $\mathbf{M}_1 \models \mathcal{T}$ means $\mu \models \mathcal{T}$ for any $\mu \in \mathbf{M}_1$ and the *theory of* $\mathbf{M}_1$ is the set $Th(\mathbf{M}_1) = \{\varphi \in \mathbf{L} \ / \ \mathbf{M}_1 \models \varphi\}$ [thus $Th(\mathbf{M}_1) = \bigcap_{\mu \in \mathbf{M}_1} Th(\mu)$]. This ambiguous use of Th and of $\models$ (applied to sets of formulas or interpretations) is usual. For any $\mathcal{T} \in \mathbf{T}$, we get $\mathcal{T} = \bigcap_{\mathcal{T}_i \in \mathbf{T},\ \mathcal{T}_i \models \mathcal{T}} \mathcal{T}_i = \bigcap_{\mathcal{C}_i \in \mathbf{T},\ \mathcal{C}_i \text{ complete},\ \mathcal{C}_i \models \mathcal{T}} \mathcal{C}_i$. The set of all the complete theories is in a natural one-to-one mapping with $\mathbf{M}$: For any $\mu \in \mathbf{M}$, $Th(\mu)$ is complete, and for any complete $\mathcal{C} \in \mathbf{T}$, $\mathbf{M}(\mathcal{C})$ is a singleton $\{\mu\} \subseteq \mathbf{M}$. For any $\mathcal{T} \subseteq \mathbf{L}$, we get $\mu \models \mathcal{T}$ iff $Th(\mu) \models \mathcal{T}$ and for any $\mathbf{M}_1 \subseteq \mathbf{M}$, $\mathbf{M}_1 \models \mathcal{T}$ iff $Th(\mathbf{M}_1) \models \mathcal{T}$.

3 PREFERENTIAL ENTAILMENTS

3.1 Preferential KLM entailment

Since their introduction [7], these kinds of preferential entailment have been extensively studied. As [4] remarks, "the use of the term *preferential* is [..] rather anarchic [...]". The situation has not really improved since these "early years", however, it is clear that now the word is not restricted to the "cumulative cases" as done in [7]. The expression "preferential entailment" was first introduced by Shoham (1988) in [12], and then regularly generalized and/or modified. The basic idea however is still the same: we consider a set of objects describing the semantics, and a binary relation $\prec$ on this set of objects. We get a "preferential semantics" in which only the objects, associated with a set of formulas, which are minimal for $\prec$, are considered. The definitions we give can be found in e.g. [7, Definitions 3.10, 3.13] ("single formula version") and [6, Definitions 4.26–29] ("theory version", only version considered here), with some modifications which have already been considered in e.g. [4, 2, 5]. These modifications are either cosmetic, or consist in dropping some special condition imposed in the original text to the relation $\prec$, since (1) we do not need these restrictions, and (2) our study can accommodate in a straightforward way these restrictions.

Definition 3.1 *A* KLM model *is a triple* $\mathbf{S} = (S, l, \prec)$*, where* S *is a set, the elements of which are called* states*,* l *is a mapping* $S \to \mathcal{P}(\mathbf{M})$ *that labels every state with a set of interpretations and* $\prec$ *is a binary relation on* S*, called a* preference relation*.*

We define a satisfaction relation $\models\!\!\!\models$*: for any* $s \in S$*,* $s \models\!\!\!\models \varphi$ *whenever* $l(s) \models \varphi$ *and, for any* $\mathcal{T} \subseteq \mathbf{L}$*,* $s \models\!\!\!\models \mathcal{T}$ *whenever* $l(s) \models \mathcal{T}$*. For any set of formulas* $\mathcal{T} \subseteq \mathbf{L}$*, we define* $S(\mathcal{T}) = \{s \in \mathbf{S} \ / \ s \models\!\!\!\models \mathcal{T}\}$ *and the set* $S_\prec(\mathcal{T})$ *of the states in* $S(\mathcal{T})$ *which are* minimal for $\prec$ *by* $S_\prec(\mathcal{T}) = \{s \in S(\mathcal{T}) \ / \ s' \prec s \text{ for no } s' \in S(\mathcal{T})\}$*.*

$S(\mathcal{T})$ is $\widehat{\mathcal{T}}$ in the original texts. Notice that, as noted by Bochman (1999) in [1], we can replace the set $l(s)$ of interpretations by a theory, precisely the theory $Th(l(s))$, and we will in fact generally prefer this formulation, where l is a mapping $S \to \mathbf{T}$ instead of $S \to \mathcal{P}(\mathbf{M})$. Also, we drop here the *consistency of states condition* $l(s) \neq \emptyset$ (or alternatively $l(s) \neq \mathbf{L}$ if we consider labelling with theories) which appears in the original definitions. As explained below, this condition is unnecessary.

The role of l is to allow *"copies of" sets of interpretations* (or alternatively *"copies" of theories*), since various states can be mapped by l to the same object.

Definition 3.2 *Let us call an* entailment relation*, any relation* $\mid\!\sim$ *in* $\mathcal{P}(\mathbf{L}) \times \mathbf{L}$*. Any entailment relation can be extended into a relation in* $\mathcal{P}(\mathbf{L}) \times \mathcal{P}(\mathbf{L})$ *by defining* $\mathcal{T} \mid\!\sim \mathcal{T}'$ *as* $\mathcal{T} \mid\!\sim \varphi$ *for any* $\varphi \in \mathcal{T}'$*.*

From any entailment relation, we can define a mapping C *from* $\mathcal{P}(\mathbf{L})$ *to itself, called an* entailment*:* $C(\mathcal{T}) = \{\varphi \in \mathbf{L} \, / \, \mathcal{T} \mid\!\sim \varphi\}$*.*

Definition 3.3 *[7] A* KLM entailment relation $\mid\!\sim^{KLM}_{\prec}$ *is defined as follows from a KLM model* $\mathbf{S}$*: for any* $\mathcal{T} \cup \{\varphi\} \subseteq \mathbf{L}$*,* $\mathcal{T} \mid\!\sim^{KLM}_{\prec} \varphi$ *whenever* $s \models\!\!\!\models \varphi$ *for any* $s \in S_\prec(\mathcal{T})$*. We write also* $\varphi \in C_{KLM}(\mathcal{T})$ *instead of* $\mathcal{T} \mid\!\sim^{KLM}_{\prec} \varphi$ *and call the entailment* C_{KLM} *a* KLM preferential entailment*, or a* KLM entailment *for short.*

Definition 3.4 *A* pre-circumscription f *(in* $\mathbf{L}$*) is an extensive (i.e.,* $f(\mathcal{T}) \supseteq \mathcal{T}$ *for any* $\mathcal{T}$*) mapping from* $\mathbf{T}$ *to* $\mathbf{T}$*. For any subset* $\mathcal{T}$ *of* $\mathbf{L}$*, we use the abbreviation* $f(\mathcal{T}) = f(Th(\mathcal{T}))$*, assimilating a pre-circumscription to a particular extensive entailment. We write* $f(\varphi)$ *for* $f(\{\varphi\}) = f(Th(\varphi))$*.*

Thus, we call here pre-circumscription any entailment which respects full logical equivalence and which is extensive. By "respects full logical equivalence", we mean that, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two logically equivalent sets of formulas [i.e. $Th(\mathcal{T}_1) = Th(\mathcal{T}_2)$], then (1) ("left side") $f(\mathcal{T}_1) = f(\mathcal{T}_2)$, and (2) ("right side") $\mathcal{T}_1 \subseteq f(\mathcal{T})$ iff $\mathcal{T}_2 \subseteq f(\mathcal{T})$. The "right side" is equivalent to "right weakening": if $\mathcal{T}_1 \models \varphi$ and $\mathcal{T}_1 \subseteq f(\mathcal{T})$, then $\varphi \in f(\mathcal{T})$.

Definition 3.5 *An entailment* C *satisfies (CT),* cumulative transitivity*, also known as "cut", if*

$$\text{for any } \mathcal{T}' \subseteq C(\mathcal{T}), \text{ we get } C(\mathcal{T} \cup \mathcal{T}') \subseteq C(\mathcal{T}).$$

Here is a characterization of KLM entailments:

Property 3.6 *[7, 8] Any KLM entailment* C_{KLM} *is a pre-circumscription satisfying* (CT)*.*

Property 3.7 *[13, 9] Any pre-circumscription satisfying* (CT) *is a KLM entailment.*

Particular KLM models can be considered. The three kinds described now originate also from [7], where no special names are given. Let $\mathbf{S} = (S, l, \prec)$ be a KLM model.

Definition 3.8 *1. If* $S = \mathcal{P}(\mathbf{M})$ *(or equivalently, under the alternative formulation in terms of theories,* $S = \mathbf{T}$*) and* $l = identity$*, then* $\mathbf{S}$ *is a* simplified *(or* unlabelled*) KLM model.*

2. If each $l(s)$ *is a singleton in* $\mathcal{P}(\mathbf{M})$ *(or equivalently a complete theory), then* $\mathbf{S}$ *is a* singular *KLM model.*

3. If $\mathbf{S}$ *is simplified and singular, then* $\mathbf{S}$ *is a* strictly singular *KLM model.*

With the unrestricted case, we get then four kinds of models, which could give rise to four kinds of KLM entailments.

We could without lack of generality restrict our attention to simplified KLM models: in the proof of Property 3.7, we can get a simplified KLM model ([9], this result could also be extracted from an independent work by Voorbraak [13]). Thus, we do not really need "states" in KLM models. However, in the case of a singular KLM model, we generally cannot suppress the states if we want to keep only singletons in the image $l(S)$ of l (as shown in a simple finite example in [7, p.193]): we cannot suppress the states without leaving this attractive particular case[2]. This feature is a good motivation for introducing states, but only in the case of singular models. Also we cannot express any KLM entailment thanks to a singular KLM model (a small finite counter-example in [7, proof of Lemma 4.5] applies

[2] Even in this case, the states can be suppressed, provided we enlarge the vocabulary of the initial language $\mathbf{L}$ in such a way that each different state gives rise to a different interpretation in the new language: this method is introduced by Costello (1998) in [3] for the cumulative and finite case

here)[3]. This means that if we start from a preference relation defined on a set of copies of sets of interpretations (or equivalently of copies of theories), then we can find an equivalent relation defined directly on the set $\mathcal{P}(\mathbf{M})$ (or $\mathbf{T}$), but we cannot always define the relation on the set $\mathbf{M}$ of interpretations or even on a set of copies of interpretations. Thus, we get three kinds of KLM preferential entailments (see their syntactical characterizations in [11]), instead of four.

A consequence of this reduction to simplified models is that any singular model is equivalent to a simplified model (generally not singular), which is not obvious from the definitions.

3.2 Preferential MAK entailment

Makinson considers an entailment more general than its KLM counterpart, with a simpler definition. The price is that this notion leaves classical consequence altogether, getting a highly non standard preferential entailment in which we can conclude $A \wedge B$ without concluding B, and in which what we conclude from $A \wedge B$ is not related to what we conclude from $\{A, B\}$. This can be useful if we want to extend the notion of preferential entailment to non classical logics. However, if we want to stay in our good old classical way of reasoning, this is rather confusing. In any case, a fair comparison with KLM definitions needs to equate the ways we want to reason at first. This is what we will do in Section 4, after giving the definitions now.

Definition 3.9 *[8] A* MAK model *is a triple* $\mathbf{S} = (S, |\!\!\equiv, \prec)$ *where* S *is a set, the elements of which are called* states, $\prec$ *is a binary relation on* S*, called a* preference relation *(till now, this is as in KLM models Definition 3.1) and where* $|\!\!\equiv$ *is any* satisfaction relation *on* S*. We define* $s|\!\!\equiv \mathcal{T}$ *by* $s|\!\!\equiv \varphi$ *for any* $\varphi \in \mathcal{T}$*. We write* $s|\!\!\equiv_\prec \varphi$ *[respectively* $s|\!\!\equiv_\prec \mathcal{T}$*] whenever* $s|\!\!\equiv \varphi$ *[respectively* $s|\!\!\equiv \mathcal{T}$*] and for no* $s' \in S$ *such that* $s' \prec s$ *we have* $s'|\!\!\equiv \varphi$ *[respectively* $s'|\!\!\equiv \mathcal{T}$*].*

A MAK entailment relation $\mid\!\sim^{MAK}_\prec$ *is defined as follows: For any set of formulas* $\mathcal{T} \cup \{\varphi\}$*,* $\mathcal{T} \mid\!\sim^{MAK}_\prec \varphi$ *whenever* $s|\!\!\equiv \varphi$ *for all* $s \in S$ *satisfying* $s|\!\!\equiv_\prec \mathcal{T}$*.*

A MAK preferential entailment*, or* MAK entailment *for short, is an entailment* C_{MAK} *defined by a MAK entailment relation:* $C_{MAK}(\mathcal{T}) = \{\varphi \in \mathbf{L} \,/\, \mathcal{T} \mid\!\sim^{MAK}_\prec \varphi\}$*.*

The names KLM model and MAK model are from [4]. What makes this short definition so powerful is that no condition is required for $|\!\!\equiv$. This makes the preferential entailment very different from what we expect for an "entailment". C_{MAK} is far from being a pre-circumscription: if $\mathcal{T}_1$ and $\mathcal{T}_2$ are classically equivalent, we do not know anything about $C_{MAK}(\mathcal{T}_2)$ when we know $C_{MAK}(\mathcal{T}_1)$. Moreover, we almost need an extensive description of all the sets $C_{MAK}(\mathcal{T})$, since they are not classical theories. If we drop the identification between a formula and its equivalence class, we can even consider logics where $C_{MAK}(A \wedge B)$ is different from $C_{MAK}(B \wedge A)$. However, we get interesting properties:

Property 3.10 *[8] MAK entailments are extensive and satisfy (CT). This implies also* idempotence*:* $C_{MAK}(C_{MAK}(\mathcal{T})) = C_{MAK}(\mathcal{T})$*.*

The significance of (CT) for such kind of "entailment" is far from being as great as when we deal with pre-circumscriptions.

Before making a comparison between MAK and KLM, let us give a few natural definitions, which extend to $(S, |\!\!\equiv)$ what is usually done with classical interpretations $(\mathbf{M}, \models)$.

[3] Again, this limitation can be overcome [10], at least in the finite case: at the price of a severe modification of the vocabulary of the initial language $\mathbf{L}$, any KLM preferential entailment can be expressed in terms of a strictly singular KLM model.

Definitions 3.11 *Let* $(S, |\!\!\equiv, \prec)$ *be a MAK model. As in Definition 3.1, for* $s \in S$*,* $\mathcal{T} \subseteq \mathbf{L}$*, and* $\varphi \in \mathbf{L}$*,* $S(\mathcal{T}) = \{s \in S / s|\!\!\equiv \mathcal{T}\}$ [and $S(\varphi) = S(\{\varphi\})$]. *We define the entailment* $Cn_{|\equiv}$ *as follows:*
$Cn_{|\equiv}(\mathcal{T}) = \{\varphi \in \mathbf{L} /$ *for any* $s \in S(\mathcal{T}), s|\!\!\equiv \varphi\}$.
We define also, for each $s \in S$*:* $Cn_{|\equiv}(s) = \{\varphi \in \mathbf{L} \,/\, s|\!\!\equiv \varphi\}$.

We get (straightforward [4]): $Cn_{|\equiv}$ is a *Tarski entailment*, i.e. it is an extensive entailment satisfying idempotence (point 1) and *monotony* (point 2): for any sets $\mathcal{T}, \mathcal{T}'$ of formulas,

1. $\mathcal{T} \subseteq Cn_{|\equiv}(\mathcal{T}) = Cn_{|\equiv}(Cn_{|\equiv}(\mathcal{T}))$.
2. If $\mathcal{T} \subseteq \mathcal{T}'$ then $Cn_{|\equiv}(\mathcal{T}) \subseteq Cn_{|\equiv}(\mathcal{T}')$.

Notice that we get: $Cn_{|\equiv}(\mathcal{T}) = \bigcap_{s \in S(\mathcal{T})} Cn_{|\equiv}(s)$.

We use the same writing $Cn_{|\equiv}$ for two different notions (cf Th in classical logic, Section 2): (a) an entailment defined by $Cn_{|\equiv}(\mathcal{T})$, and (b) a notion of "theory of a state" defined by $Cn_{|\equiv}(s)$. A justification for this is that $Cn_{|\equiv}(s)$ is indeed a *theory in the meaning of* $Cn_{|\equiv}$: for any $s \in S$, $Cn_{|\equiv}(Cn_{|\equiv}(s)) = Cn_{|\equiv}(s)$.

Proof: $Cn_{|\equiv}(s) \subseteq Cn_{|\equiv}(Cn_{|\equiv}(s))$ since $Cn_{|\equiv}$ is a Tarski entailment. $Cn_{|\equiv}(Cn_{|\equiv}(s)) = \bigcap_{s' \in S(Cn_{|\equiv}(s))} Cn_{|\equiv}(s')$ by definition of $Cn_{|\equiv}(\mathcal{T})$, and $s \in S(Cn_{|\equiv}(s))$ [i.e. $s|\!\!\equiv Cn_{|\equiv}(s)$], thus $Cn_{|\equiv}(Cn_{|\equiv}(s)) \subseteq Cn_{|\equiv}(s)$. □

Here is another result [4]: For any $\mathcal{T} \subseteq \mathbf{L}$, $Cn_{|\equiv}(\mathcal{T}) \subseteq C_{MAK}(\mathcal{T})$. Notice that C_{MAK} is not a Tarski entailment: it falsifies monotony.

4 RELATING MAK AND KLM ENTAILMENTS

4.1 The MAK entailments which are KLM ones

MAK notion encompasses KLM notion, as noticed in [8, 4]:

Property 4.1 *For each KLM model* $\mathbf{S} = (S, l, \prec)$ *(defining a KLM entailment* C_{KLM}*), there exists a MAK model* $(S, |\!\!\equiv, \prec)$ *such that its associated MAK entailment* C_{MAK} *is equal to* C_{KLM}*.*

The states and the relation $\prec$ are unmodified, $|\!\!\equiv$ is defined by: $s|\!\!\equiv \varphi$ whenever $l(s) \models \varphi$ [4]. We get then clearly $\mid\!\sim^{MAK}_\prec = \mid\!\sim^{KLM}_\prec$.

Here is a characterization of the subclass of MAK models which can be translated into KLM models.

Theorem 4.2 *A MAK model* $\mathbf{S} = (S, |\!\!\equiv, \prec)$ *gives rise to a MAK entailment* C_{MAK} *which is equal to some KLM entailment* C_{KLM} *iff the MAK entailment* C_{MAK} *is a pre-circumscription.*

Proof: The condition is necessary from Prop. 3.6. MAK entailments satisfy (CT) from Prop. 3.10, thus Prop. 3.7 gives the result. □

Thus, what lacks to MAK entailment in order to be a KLM entailment is exactly the full preservation of logical equivalence.

This result is more a characterization of the subclass of the MAK entailments which can be turned into KLM entailments than a characterization of the subclass of MAK models which can be turned into KLM models. We will now get an easier and more direct property, which can be checked directly on the MAK model.

4.2 A subclass of MAK models which are KLM models

For any KLM entailment, we have $Th(\mathcal{T}) \subseteq C_{KLM}(\mathcal{T})$. This has been taken into account in [4] for describing a subclass of the MAK models which (1) can be turned into KLM models and (2) is powerful enough to give all the KLM entailments. However, in [4], the description of this subclass is needlessly complex. We will describe now a simpler and more general subclass satisfying (1) and (2).

Definitions 4.3 *1. A MAK model* $\mathbf{S} = (S, |\!\!\equiv, \prec)$ *is* supra classical *whenever we get* $Th(Cn_{\models}(s)) = Cn_{\models}(s)$ *for any state* s *in* S. *This means that the "world" associated to each state is classically deductively closed (i.e. is a classical theory).*

2. An entailment C *is* supra classical *if it satisfies* $Th(\mathcal{T}) \subseteq C(\mathcal{T})$ *for any* $\mathcal{T} \subseteq \mathbf{L}$.

The first definition is independent of the preference relation $\prec$. Voorbraak [13] calls "**L**-faithful" our supra classical models.

For the second definition, notice that any pre-circumscription is a supra classical entailment.

Lemma 4.4 *If a MAK model* $\mathbf{S} = (S, \models, \prec)$ *is such that the entailment* $Cn_{\models}$ *is supra classical, then* $\mathbf{S}$ *is supra classical.*

Proof: (1) $Cn_{\models}(s) \subseteq Th(Cn_{\models}(s))$ since $\mathcal{T} \subseteq Th(\mathcal{T})$.

(2) $Th(Cn_{\models}(s)) \subseteq Cn_{\models}(Cn_{\models}(s))$ by hypothesis, and we already know that we have in any case $Cn_{\models}(Cn_{\models}(s)) = Cn_{\models}(s)$, which establishes $Th(Cn_{\models}(s)) = Cn_{\models}(s)$. □

Lemma 4.5 *If a MAK model* $\mathbf{S} = (S, \models, \prec)$ *is supra classical, then the (Tarski) entailment* $Cn_{\models}$ *and the (preferential) MAK entailment* C_{MAK} *defined by* $\mathbf{S}$ *are pre-circumscriptions.*

Proof: (1) Let $\varphi \in Th(\mathcal{T})$ and $s \in S(\mathcal{T})$, then $\varphi \in Cn_{\models}(s)$ by supra classicality of $\mathbf{S}$. Thus, $\varphi \in \bigcap_{s \in S(\mathcal{T})} Cn_{\models}(s) = Cn_{\models}(\mathcal{T})$: $Cn_{\models}$ is supra classical. From $Cn_{\models}(\mathcal{T}) = \bigcap_{s \in S(\mathcal{T})} Cn_{\models}(s)$ we get also $Cn_{\models}(\mathcal{T}) \in \mathbf{T}$ since each $Cn_{\models}(s)$ is in $\mathbf{T}$. Since by Definition 3.9 we get $C_{MAK}(\mathcal{T}) = \bigcap_{s \in S_{\prec}(\mathcal{T})} Cn_{\models}(s)$ (where the subset $S_{\prec}(\mathcal{T})$ of $S(\mathcal{T})$ is defined exactly as in Definition 3.1 for KLM models), we get a fortiori $\varphi \in C_{MAK}(\mathcal{T}) \in \mathbf{T}$.

(2) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two equivalent sets, then, by supra classicality of $\mathbf{S}$, for each $s \in S$, we have $s|\!\!\equiv \mathcal{T}_1$ iff $s|\!\!\equiv \mathcal{T}_2$, i.e. we have $S(\mathcal{T}_1) = S(\mathcal{T}_2)$. A fortiori we get then $S_{\prec}(\mathcal{T}_1) = S_{\prec}(\mathcal{T}_2)$. Thus we get $Cn_{\models}(\mathcal{T}_1) = Cn_{\models}(\mathcal{T}_2)$ and $C_{MAK}(\mathcal{T}_1) = C_{MAK}(\mathcal{T}_2)$ by the definitions of $Cn_{\models}(\mathcal{T})$ and $C_{MAK}(\mathcal{T})$ respectively. □

We have established:

Property 4.6 *1. A MAK model* $\mathbf{S} = (S, \models, \prec)$ *is supra classical iff the (Tarski) entailment* $Cn_{\models}$ *that it defines is supra classical, iff the entailment* $Cn_{\models}$ *is a pre-circumscription.*

2. If a MAK model is supra classical, then the (preferential) MAK entailment C_{MAK} *that it determines is a pre-circumscription.*

We are now in position to establish our second main result:

Theorem 4.7 *1. If a MAK model* $\mathbf{S}$ *is supra classical, then the MAK entailment* C_{MAK} *that it determines is a KLM entailment* C_{KLM}. *Precisely, if* $\mathbf{S} = (S, \models, \prec)$ *is supra classical, there exists a KLM model* $\mathbf{S}' = (S, l, \prec)$ *with* S *and* $\prec$ *unmodified, such that the MAK entailment defined by* $\mathbf{S}$ *is the KLM entailment defined by* $\mathbf{S}'$.

2. Any KLM preferential entailment C_{KLM} *is equal to a MAK entailment defined by a supra classical MAK model. Precisely, if* $\mathbf{S}$=$(S,l,\prec)$, *there exists a supra classical MAK model* $\mathbf{S}' = (S, \models, \prec)$ *with* S *and* $\prec$ *unmodified, such that the KLM entailment defined by* $\mathbf{S}$ *is the MAK entailment defined by* $\mathbf{S}'$.

Proof: (1) Theorem 4.2 and Property 4.6-2 give the first sentence.

Let $\mathbf{S} = (S, |\!\!\equiv, \prec)$ be a supra classical MAK model. We get a KLM model as follows: keeping S and $\prec$ unmodified, we define l as the mapping $S \to \mathbf{T}$ by taking $l(s) = Cn_{\models}(s)$. It is immediate to see that the KLM entailment C_{KLM} is equal to C_{MAK}.

(2) We have a constructive proof already: It suffices to see the construction given in Property 4.1: it is clear from the definitions that the MAK model obtained there is supra classical since we have already noticed that each $l(s)$ in Definition 3.1 can be equated to a classical theory. □

Notice that the "cumulative" version of this theorem could also have been obtained from some results in an earlier independent work by Voorbraak [13]. Rather strangely, Voorbraak does not enounce these results, referring to [4] for further results on the subject.

Thus, we get characterization results and constructive passages simpler and easier than those given in [4]. However, our results are slightly more general (see why in note 5): the subclass of the MAK models considered here is slightly greater than the subclass considered by Dix and Makinson since they consider a strict subclass of the MAK models which can be "amplified" (in their terms). It is easy to see, from [4] together with our results, that the class of the MAK models which can be "amplified" coincide with the class of the supra classical MAK models. Moreover, our comparison does not need to consider a third intermediate (between $Cn_{\models}$ and C_{MAK}) non classical entailment[4], which plays an important role in the results of [4], but which complicates the direct comparison between KLM and MAK preferential entailments. This simplification comes mainly from our results about supra classical MAK models. And the condition that each $Cn_{\models}(s)$ must be a theory is easily checked, without the need to compute the associated MAK entailment or to introduce a third non classical entailment.

If we are concerned only by those MAK entailments which respect full logical equivalence, we can restrict our attention to a yet narrower class of MAK models. Indeed, we have seen why in this case we can restrict our attention to the easily described class of supra classical MAK models. Now, since we can consider only the simplified version of KLM models, our passages between MAK models and KLM models show that we can also require a *unicity of states* condition for MAK models. By "unicity of states", we mean that, for any different $s, s' \in S$, the "worlds" $Cn_{\models}(s)$ and $Cn_{\models}(s')$ corresponding to these two states are different. The class of the supra classical MAK models satisfying unicity of states is powerful enough to generate all the MAK entailments which are pre-circumscriptions. Let us describe now the analogous of the singular and the strictly singular KLM models in terms of MAK models.

Definition 4.8 *A MAK model* $\mathbf{S} = (S, |\!\!\equiv, \prec)$ *is* classical *if the "worlds"* $Cn_{\models}(s)$ *are (classical) complete theories, for any* $s \in S$.

Remark 4.9 • *A MAK model is supra classical iff the satisfaction relation* $\models$ *respects the binary connector* $\wedge$: *for each* $s \in S$, *we have*

$$s \models \varphi_1 \wedge \varphi_2 \quad \textit{iff} \quad s \models \varphi_1 \textit{ and } s \models \varphi_2. \quad (R_\wedge)$$

• • *A MAK model is classical iff it is supra classical and* $\models$ *respects the negation* $\neg$:

$$s \not\models \varphi \quad \textit{iff} \quad s \models \neg\varphi. \quad (R_\neg)$$

Proof: • If each $Cn_{\models}(s)$ is in $\mathbf{T}$, then, since $\{\varphi_1, \varphi_2\} \equiv \{\varphi_1 \wedge \varphi_2\}$, we get $\{\varphi_1, \varphi_2\} \subseteq Cn_{\models}(s)$ iff $\{\varphi_1 \wedge \varphi_2\} \subseteq Cn_{\models}(s)$. Conversely, let us suppose $(R_\wedge)$. Then, if $\{\varphi_i\}_{i \in I} \subseteq Cn_{\models}(s)$ and

[4] For readers familiar with [4], let us notice that an immediate consequence of our results is that, even in the exact framework and formulation considered in [4, main theorem], the condition (3a) given there ($s|\!\!\equiv \varphi$ and $s|\!\!\equiv \varphi'$ implies $s \models \varphi \wedge \varphi'$) is redundant.

$\{\varphi_i\}_{i \in I} \models \varphi$, by compactness of $\models$ there exists a finite $J \subseteq I$ such that $\{\varphi_i\}_{i \in J} \models \varphi$, i.e. $\bigwedge_{i \in J} \varphi_i \models \varphi$, i.e. $\bigwedge_{i \in J} \varphi_i \equiv (\bigwedge_{i \in J} \varphi_i) \wedge \varphi$, thus, by $(R_\wedge)$, $\varphi \in Cn_{\models\!\equiv}(s)$. Remind that we identify a formula with its equivalence class. Makinson does not always make this assumption in [8], thus, his original formalism is slightly more general than the version given in the present text. However, since with KLM entailments a formula can always be replaced by an equivalent formula, we must make this assumption when we want to compare the two formalisms: If this assumption is not made till the beginning (as in this text), then it must be added, e.g. by requiring in Definition 4.3-1 that $\models$ is standard. Notice that Definition 4.3-1 as it stands implies that two formulas equivalent (for $\models$) are always in the same sets $Cn_{\models\!\equiv}(s)$, thus are "equivalent for $\models\!\equiv$".

• • A theory $Cn_{\models\!\equiv}(s)$ is complete iff [$\varphi \in Cn_{\models\!\equiv}(s)$ iff $\neg\varphi \notin Cn_{\models\!\equiv}(s)$], i.e. iff $\models\!\equiv$ satisfies $(R_\neg)$. □

A MAK model is classical iff $\models\!\equiv$ respects all the logical connectors. For instance, it is immediate to see that $(R_\wedge)$ and $(R_\neg)$ imply $(R_\vee)$:

$$s \models\!\equiv \varphi_1 \vee \varphi_2 \quad \text{iff} \quad s \models\!\equiv \varphi_1 \ \text{ or } \ s \models\!\equiv \varphi_2. \qquad (R_\vee)$$

Classical MAK models correspond to singular KLM models while the classical MAK models which respect unicity of states correspond to the strictly singular KLM models.

4.3 Coming back to the original KLM entailments

This work applies also to cases where special conditions are required for the models. We think that the simplicity and the naturalness of our translation is a first serious indication for this. Let us consider the original definitions.

Definitions 4.10 *1. A* consistent KLM model*, is such that each state is* consistent*, meaning that $l(s)$ is consistent.*

2. A KLM model $\mathbf{S} = (S, l, \prec)$ is smooth *(*stoppered *in [8]) if, for each $\mathcal{T} \subseteq \mathbf{L}$ and $s \in S(\mathcal{T}) - S_\prec(\mathcal{T})$, there exists $s' \in S_\prec(\mathcal{T})$ such that $s' \prec s$ ("minoration by a minimal state").*

[7] considers only the KLM models which are consistent and smooth. The authors consider that *cumulative monotony* (CM) [if $\mathcal{T}' \subseteq C(\mathcal{T})$, then $C(\mathcal{T}) \subseteq C(\mathcal{T} \cup \mathcal{T}')$] is as important as (CT), and they only care of *cumulative entailments*, which satisfy (CT) and (CM). They give the following characterization [7, Theorem 3.25]:

> A pre-circumscription C is cumulative iff it is a KLM entailment defined by a smooth and consistent KLM model.

This characterization result also holds without the consistency condition, which confirms our opinion that, for KLM entailments, the requirement that $l(s)$ must be consistent is needless[5].

We get, with the KLM models as defined here, a first modification of KLM characterization (the proof is an easy modification of the proof of the original characterization [7], moreover this result has already appeared as [8, Observation 3.4.5] and [13, Proposition 5.4]):

> A pre-circumscription C is cumulative iff it is a KLM entailment defined by a smooth KLM model.

We can go even further, by requiring that the KLM model is a *simplified KLM model* (see the companion paper published under the same tittle as the present paper in the Workshop NMR 2002, § 4.3).

[5] For readers familiar with [4], let us remind that the main theorem of Dix and Makinson adds to (3a) (see note 4) a "consistency of states" condition (3b). This condition is necessary because they disallow inconsistent states in KLM models, following [7]. Since inconsistent states are not a real problem, this condition, which restricts the class of the MAK models concerned, can be suppressed without modifying the results about the entailments.

5 CONCLUSION AND PERSPECTIVES

We have shown that the notions of preferential entailment as defined by Kraus, Lehmann and Magidor and as defined by Makinson are much closely related than was supposed before. Indeed, these two notions coincide exactly in all the cases where they can coincide, that is when the underlying logic respects classical equivalence. Moreover, we have shown that a similar result holds also for the respective models defining the two notions. It was already known that any KLM model could easily be turned into a MAK model. We have exhibited a natural subclass of the MAK models which can, exactly as easily, be turned into a KLM model. The subclass obtained here is slightly greater, and is much easier to describe, than what was previously known. And this subclass of models is "complete": it generates all the KLM preferential entailments. This subclass is the class of the MAK models for which all the states have a "classical" behavior: the set of formulas they satisfy is closed for classical deduction. This subclass is the most natural class to consider. Indeed, this is the class such that, for any preference relation $\prec$, we are certain from the beginning that the MAK preferential entailment generated has a classical behavior with respect to logical equivalence. Some MAK models outside this class can give rise to a KLM entailment, but these models can easily be turned into classically behaved ones.

Our results apply to important particular subclasses of KLM models: (1) the models which are simplified in that the labelling mapping l is needless, (2) the "smooth"models (corresponding to cumulative preferential entailments).

As (non trivial) future work, let us remark that these results should help further study on the subject, since they show that this kind of preferential entailment is not as "cumbersome" as it is qualified even in the founding paper [7]. Even automatic computation could take advantage from these results, since the models considered here have nice properties, which, hopefully, could help designing new kinds of "preferential entailments demonstrators".

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